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2002 J. Phys. A: Math. Gen. 35 L613

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LETTER TO THE EDITOR

Leading off-diagonal approximation for the spectral form factor for uniformly hyperbolic systems

Martin Sieber

School of Mathematics, University of Bristol, Bristol BS8 1TW, UK

E-mail: m.sieber@bristol.ac.uk

Received 2 August 2002

Published 8 October 2002

Online at stacks.iop.org/JPhysA/35/L613

Abstract

We consider the semiclassical approximation to the spectral form factor $K(\tau)$ for two-dimensional uniformly hyperbolic systems with time-reversal symmetry, and derive the first off-diagonal correction for small τ . The result agrees with the τ^2 -term of the form factor for the GOE random matrix ensemble.

PACS numbers: 03.65.Sq, 05.45.Mt

A fundamental characteristic of quantum systems that are chaotic in their classical limit is universality. It is observed that diverse systems behave identically when statistics of energy levels or wavefunctions are considered, provided that they have the same symmetries. These universal statistics agree with those of random matrix theory, i.e. with the statistics of eigenvalues and eigenvectors of large random matrices [1]. Support for this random matrix hypothesis comes from a large number of numerical and experimental investigations which have been carried out on a great variety of systems [2]. However, it remains an open question to understand the origin of this universality, and its relation to the underlying classical dynamics.

One theoretical approach by which such an understanding may be attempted is the semiclassical method. Semiclassical approximations are asymptotically valid in the limit $\hbar \rightarrow 0$ where universality is expected to hold. Moreover, they directly connect quantum properties with properties of the corresponding chaotic classical system. They have been applied in particular to statistical distributions of the energy levels which are bilinear in the density of states, one example being the spectral form factor $K(\tau)$. One of the successes of the semiclassical approach has been to show that the spectral statistics do indeed agree with the random matrix statistics in the limit of long-range correlations; specifically the correct leading order behaviour of $K(\tau)$ as $\tau \rightarrow 0$ has been derived [3].

An extension of this result requires knowledge of correlations between different periodic orbits [4]. The relevant mechanisms by which periodic orbits are correlated have to be identified, and the contributions of correlated orbits to the spectral form factor have to be evaluated. Based on an analogy with disordered systems [5] and with diffractive corrections [6]

it has been suggested that the next term in the expansion of $K(\tau)$ for small τ originates from ‘two-loop orbits’: orbits that have a self-intersection with small crossing angle and neighbouring orbits without self-intersection [7]. There is strong numerical evidence that in systems with time-reversal symmetry these orbit pairs indeed yield the next order term in agreement with the expectation based on random matrix theory.

In the following we present a derivation of the next to leading order term in the expansion of the spectral form factor for small τ . We evaluate analytically the contributions of the two-loop orbits to the form factor for uniformly hyperbolic systems with time-reversal symmetry, and we show that the result indeed agrees with random matrix theory. The calculation makes clear the properties of classical trajectories which are responsible for the universal result. The approach presented here has already been generalized to quantum graphs for which the corresponding term of the form factor has also been obtained [8].

We consider the spectral form factor, which is defined as the Fourier transform of the two-point correlation function of the density of states

$$K(\tau) = \int_{-\infty}^{\infty} \frac{d\eta}{\bar{d}(E)} \left\langle d_o \left(E + \frac{\eta}{2} \right) d_o \left(E - \frac{\eta}{2} \right) \right\rangle_E e^{2\pi i \eta \tau \bar{d}(E)} \quad (1)$$

where the density of states $d(E) = \sum_n \delta(E - E_n)$ is divided into a mean part $\bar{d}(E)$ and an oscillatory part $d_o(E)$. For systems with time-reversal symmetry, or more generally an anti-unitary symmetry, it is expected that the form factor agrees in the semiclassical limit ($\hbar \rightarrow 0$) with that of the Gaussian orthogonal ensemble (GOE) of random matrix theory which has the expansion

$$K^{\text{GOE}}(\tau) = 2\tau - 2\tau^2 + \mathcal{O}(\tau^3) \quad \text{as } \tau \rightarrow 0. \quad (2)$$

The semiclassical approximation for the form factor is obtained by inserting Gutzwiller’s trace formula for the density of states into (1) and evaluating the integral in leading order of \hbar . The result is an approximation in terms of a double sum over all periodic orbits of the classical system

$$K(\tau) \approx \frac{1}{h\bar{d}(E)} \left\langle \sum_{\gamma, \gamma'} A_\gamma A_{\gamma'}^* e^{\frac{i}{\hbar}(S_\gamma - S_{\gamma'})} \delta \left(T - \frac{T_\gamma + T_{\gamma'}}{2} \right) \right\rangle_E \quad (3)$$

where $\tau = T/(h\bar{d}(E))$ and $h = 2\pi\hbar$. Furthermore, A_γ is an amplitude, generally complex-valued, which depends on the stability and the Maslov index of the periodic orbit γ , and S_γ and T_γ are its action and period.

The double sum in (3) runs over all possible pairings of periodic orbits. However, most of these pairs do not contribute in the semiclassical limit. Periodic orbits which are located in different regions in phase space are uncorrelated, and when summed over, the contributions from different pairs cancel each other. It is expected that the relevant semiclassical contributions come from a relatively small number of pairs of orbits which are correlated. The key problem is then to identify the mechanism which is behind these correlations.

The basic assumption we make is that only those periodic orbits which are almost everywhere close to one another, or to the time reverse of the other orbit, are correlated [7]. In order for two orbits to be different but nevertheless close they must have special forms which can be constructed in the following way. The orbits are composed of different segments during which one orbit follows very closely the other orbit (or its time reverse). However, the orbits can differ in the way in which the segments are connected.

The two simplest possibilities are shown in figure 1. If orbits are composed of only one segment, then the two ends can be connected in only one way. It then follows that the two

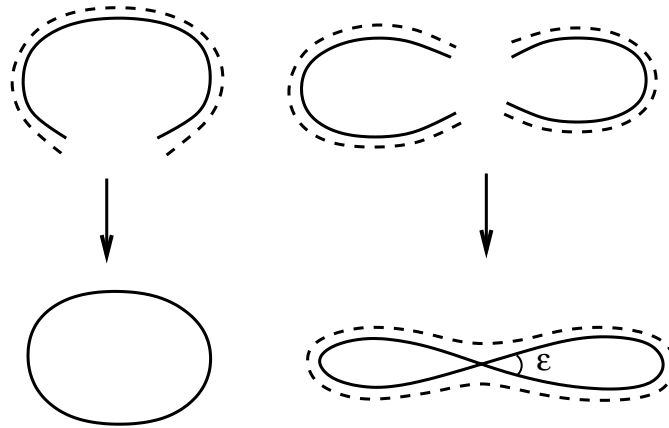


Figure 1. The pairs of orbits considered here consist of different segments. In each segment one orbit is very close to the other (or its time reverse), but they can differ in the way the segments are connected.

neighbouring orbits are either identical or one is the time reverse of the other. Including only these pairs in the double sum corresponds to the diagonal approximation, which yields the correct leading order behaviour $K(\tau) \sim 2\tau$ as $\tau \rightarrow 0$ [3].

Two segments, on the other hand, can be connected in two ways, leading to orbits with or without self-intersection at the connection point, as shown in figure 1. In order for these pairs to exist and to be close, the crossing angle ε has to be small. Then it can be shown in a linearized approximation that one orbit is indeed in the neighbourhood of the other.

In the following we evaluate the contributions of such pairs of orbits to the spectral form factor. In order to avoid further assumptions and to keep the calculations simple we restrict attention now to systems with uniformly hyperbolic dynamics; specifically we consider the representative example of the geodesic motion on Riemann surfaces with constant negative curvature [9]. Then the quantities A_γ , S_γ and T_γ in (3) depend only on the length of an orbit, and A_γ is positive. We assume that the systems have no further symmetries and are non-arithmetic so that the typical degeneracy of a length of a periodic orbit is 2. For these systems the action difference for the pairs of orbits being considered here has been calculated in the linearized approximation for small crossing angle ε in [7] and is given by

$$\Delta S(\varepsilon) \approx \frac{p^2 \varepsilon^2}{2m\lambda} \quad (4)$$

where λ is the Lyapunov exponent of the system, and p and m are momentum and mass of the particle, respectively.

The sum over these pairs of orbits can be evaluated by summing over all self-intersections of periodic orbits with small crossing angle ε , because for every such self-intersection there exists a neighbouring periodic orbit with action difference $\Delta S(\varepsilon)$. The self-intersections are determined by introducing a function which selects them. This is done in the following way. A self-intersection of a periodic orbit with period T divides the orbit into two loops. It can be characterized by the crossing angle ε and the total time t along the shorter of the two loops, $t \leq T/2$. Furthermore, we introduce an angle variable ϕ that specifies the direction of the velocity, and a variable t' that measures the time along a periodic orbit. If at any time t' along a periodic orbit $\mathbf{q}(t' + t) = \mathbf{q}(t')$ and $\phi(t' + t) = \phi(t') - \pi + \varepsilon$, then this periodic orbit has

a self-intersection with opening angle ε , and traversing the corresponding shorter loop takes time t .

Correspondingly, we can express the contribution from pairs of the two-loop orbits to the form factor as

$$K^{(2)}(\tau) = \frac{4}{\hbar \bar{d}(E)} \operatorname{Re} \int_{-\pi}^{\pi} d\varepsilon \int_0^{T/2} dt \sum_{\gamma} A_{\gamma}^2 e^{\frac{i}{\hbar} \Delta S(\varepsilon)} \delta(T - T_{\gamma}) \int_0^T dt' f_{\varepsilon,t}(\mathbf{q}(t'), \mathbf{p}(t')) \quad (5)$$

where the function $f_{\varepsilon,t}$ is given by

$$f_{\varepsilon,t}(\mathbf{q}(t'), \mathbf{p}(t')) = |J| \delta(\phi(t'+t) - \phi(t') + \pi - \varepsilon) \delta(\mathbf{q}(t'+t) - \mathbf{q}(t')). \quad (6)$$

Here $|J| = v^2 |\sin \varepsilon| / \sqrt{g}$ is the Jacobian for the transformation from the arguments of the three delta functions to the three integration variables, where v is the speed of the particle and g is the determinant of the metric tensor. The three integrals give a contribution each time that t' is at the beginning of a loop with time t and opening angle ε . The choice of the limits of the integral over ε is not important since the main contribution in the semiclassical limit $\hbar \rightarrow 0$ comes from the asymptotic behaviour of the integrand at $\varepsilon = 0$. In (5) the amplitudes and the periods of the neighbouring orbits were set to be equal since the difference does not contribute to the leading semiclassical order.

One of the important properties of long periodic orbits is their uniform distribution on the energy surface in phase space. It implies that the average of a given phase space function $f(\mathbf{q}, \mathbf{p})$ along all periodic orbits of a certain period T can be replaced, in the limit $T \rightarrow \infty$, by an average of this function over the energy surface in phase space [10]. More accurately, the following asymptotic relation holds as $T \rightarrow \infty$,

$$\sum_{\gamma} |A_{\gamma}|^2 \delta(T - T_{\gamma}) \int_0^T dt' f(\mathbf{q}(t'), \mathbf{p}(t')) \sim \frac{T^2}{\Sigma(E)} \int d^2q d^2p \delta\left(E - \frac{p^2}{2m}\right) f(\mathbf{q}, \mathbf{p}) \quad (7)$$

where $\Sigma(E)$ is the volume of the energy surface in phase space.

Relation (5) is in the form in which this property of the periodic orbits can be applied. The semiclassical limit $\hbar \rightarrow 0$ is performed with the condition that $\tau/\hbar \rightarrow \infty$. The mean density of states being $\bar{d}(E) \sim \Sigma(E)/(2\pi\hbar)^2$, this implies that $T \rightarrow \infty$ and thus the leading order semiclassical behaviour arises from the large T behaviour. Applying the uniformity of the periodic orbit distribution and performing the integral over the energy delta-function one obtains

$$K^{(2)}(\tau) \sim \frac{4p^2 T^2}{m\hbar \bar{d}(E)} \operatorname{Re} \int_{-\pi}^{\pi} d\varepsilon e^{\frac{ip^2 \varepsilon^2}{2m\hbar \kappa}} \sin|\varepsilon| \int_0^{T/2} dt p_E(\varepsilon, t) \quad (8)$$

where

$$p_E(\varepsilon, t) = \int \frac{d^2q_0 d\phi_0}{\Sigma(E)} \delta(\mathbf{q}(t) - \mathbf{q}_0) \delta(\phi(t) - \phi_0 + \pi - \varepsilon) \quad (9)$$

and $\mathbf{q}(t)$ and $\phi(t)$ are the coordinates of a particle at time t , whose initial conditions at $t = 0$ are specified by \mathbf{q}_0 , ϕ_0 and energy E .

The quantity $p_E(\varepsilon, t)$ has a direct classical interpretation. It is the probability density for a particle with energy E to return after time t to its starting point with a velocity that deviates from the initial velocity by an angle $\varepsilon - \pi$. In the same way as for the diagonal approximation, one thus finds that the periodic orbit sum is related to a transition probability density in phase space [11].

Our aim is to determine the leading order behaviour of (8) as $\hbar \rightarrow 0$ which, as remarked above, depends on the long-time behaviour of $p_E(\varepsilon, t)$. For long times $p_E(\varepsilon, t)$ approaches one over the volume of the energy shell in phase space, because the particle is equally likely

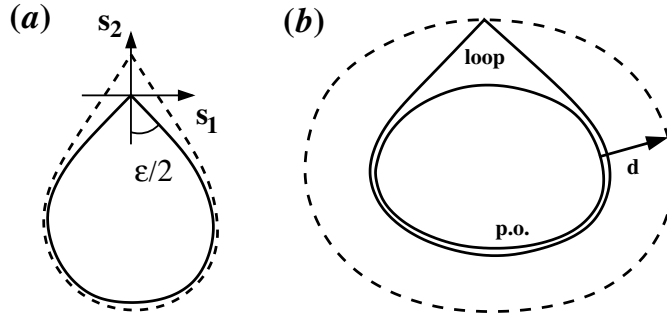


Figure 2. (a) A loop with opening angle ε (full line), and the local coordinate system at its starting point. (b) Loops with the same opening angle ε whose traversal takes time t form a continuous family and have their starting points on a curve of constant distance d from a periodic orbit.

to be found anywhere on the energy shell, i.e. $p_E(\varepsilon, t) \sim 1/\Sigma(E)$ as $t \rightarrow \infty$. Inserting this into (8) and applying the method of stationary phase yields

$$\frac{p^2 \tau^3 \Sigma(E)}{m\pi^2 \hbar^2} \operatorname{Re} \int_0^\infty d\varepsilon \exp\left(\frac{ip^2 \varepsilon^2}{2m\hbar\lambda}\right) \varepsilon = 0 \quad (10)$$

and so the leading order term as $\hbar \rightarrow 0$ vanishes. This implies that one has to take into account the next order terms. A closer analysis of (5) shows that the important term to consider is the next to leading order behaviour of $p_E(\varepsilon, t)$ as $t \rightarrow \infty$. Quite surprisingly, the two-loop contribution does not originate from the ergodic limit of the probability density $p_E(\varepsilon, t)$ but from the approach to this limit.

We have to consider $p_E(\varepsilon, t)$ in more detail. In the following we examine its properties for arbitrary values of ε , although we finally need only its behaviour for small ε . The quantity $p_E(\varepsilon, t)$ is a classical transition probability density and can be expressed in terms of classical trajectories. These trajectories are all time t loops with opening angle ε . Consider one such loop as shown in figure 2(a). Every point in the vicinity of its starting point is the starting point of another loop, one example being shown by the dashed line. To determine how angle ε and time t change with the initial point we introduce a local coordinate system (see figure 2(a)) and linearize the motion in the vicinity of the loop. The result is

$$v dt = 2 \cos \frac{\varepsilon}{2} ds_2 \quad v d\varepsilon = -2\lambda \sin \frac{\varepsilon}{2} \tanh \frac{\lambda t}{2} ds_2. \quad (11)$$

One finds that angle and time change only in the s_2 direction, but not in the s_1 direction. This is a particular property of the uniformly hyperbolic dynamics. After integrating equations (11) one arrives at the following conclusion. The loops with fixed ε and t form continuous one-parameter families. All the initial points of the loops within a family lie on a curve which has a constant distance (denoted by d) from a periodic orbit as shown schematically in figure 2(b). The relation between the loops and the periodic orbit is given by

$$\cosh \frac{\lambda t}{2} \sin \frac{|\varepsilon|}{2} = \cosh \frac{\lambda t_0}{2} \quad (12)$$

where t_0 is the period of the periodic orbit. It is a remarkable property that any loop is uniquely related to a periodic orbit into which it can be continuously deformed through a series of other loops. Put another way, this implies that any self-intersection of any arbitrary classical trajectory is uniquely related to a periodic orbit, because a self-intersection is the initial point of a loop.

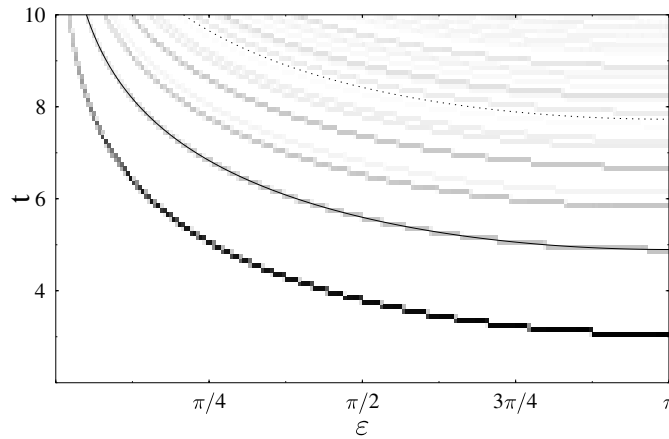


Figure 3. Numerical result of the search for loops with opening angle ε and time t . Grey scales are proportional to the number of loops found in bins in the (ε, t) -plane.

We examined this property numerically. We chose a large number of long random trajectories on a Riemann surface with constant negative curvature [12] and recorded all their self-intersections. For every self-intersection a point is plotted in the (ε, t) -plane, where ε and t are the opening angle and traversal time, respectively, of the corresponding loop. The result is shown in figure 3. As expected, the points form continuous lines that start at the periods of the periodic orbits (the t -values at $\varepsilon = \pi$). One can observe a logarithmic divergence of the curves at $\varepsilon = 0$ which is implied by equation (12). The full line in figure 3 is an evaluation of equation (12) for the second family of loops, and it is found to be in perfect agreement with the numerical result.

We continue by expressing $p_E(\varepsilon, t)$ in terms of the classical trajectories. By evaluating the integrals over the delta-functions in (9), $p_E(\varepsilon, t)$ can be written as a sum over all families of loops with opening angle ε , which are labelled by ξ in the following. Alternatively, by using relation (12), $p_E(\varepsilon, t)$ can also be expressed in terms of the periodic orbits labelled by ξ_0 ,

$$\begin{aligned} p_E(\varepsilon, t) &= \frac{1}{\Sigma(E)} \sum_{\xi} \frac{T_{\xi_0} \cosh(\lambda d/v) \delta(t - T_{\xi})}{\sin|\varepsilon/2|(\text{Tr } M_{\xi} - 2)} \\ &= \frac{1}{\Sigma(E)} \sum_{\xi_0} \frac{T_{\xi_0} \delta(t - T_{\xi})}{\sqrt{(\text{Tr } M_{\xi_0} - 2) (\text{Tr } M_{\xi_0} - 2 + 4 \cos^2 \frac{\varepsilon}{2})}} \end{aligned} \quad (13)$$

where M_{ξ} and M_{ξ_0} denote the stability matrices. We remark that a further use of equation (12) yields

$$p_E(\varepsilon, t) = p_E(\pi, t_0). \quad (14)$$

This means that the distribution $p(\varepsilon, t)$ is identical to the return probability density $p(\pi, t_0)$ at a shifted time t_0 , the relation between t and t_0 being given by equation (12).

Equation (13) is now applied to find the next to leading order behaviour of the time integral over $p_E(\varepsilon, t)$ as $t \rightarrow \infty$. We assume that from a certain time $T_0(\varepsilon)$ on we can replace $p_E(\varepsilon, t)$ by its ergodic limit $1/\Sigma(E)$. This time $T_0(\varepsilon)$ is chosen to have the same ε -dependence as the time of the families of loops (like, for example, the dashed line in figure 3). Thus $T_0(\varepsilon)$ is related to $T_0(\pi)$ by an equation identical to that between t and t_0 (equation (12)). For

$t < T_0(\varepsilon)$ we replace $p_E(\varepsilon, t)$ by its exact form, equation (13). The approximation can be made asymptotically exact by letting $T_0(\pi) \rightarrow \infty$ as $T \rightarrow \infty$. We find

$$\begin{aligned} \int_0^{T/2} dt p_E(\varepsilon, t) &\sim \int_{T_0(\varepsilon)}^{T/2} dt \frac{1}{\Sigma(E)} + \sum_{T_{\xi_0} < T_0(\pi)} B_{\xi_0}(\varepsilon) \\ &= \frac{T/2 - T_0(\varepsilon)}{\Sigma(E)} + \text{const} + \mathcal{O}(\varepsilon^2) \end{aligned} \quad (15)$$

where here and in the following constant denotes independence of ε . In the semiclassical limit only the asymptotic behaviour of equation (15) as $\varepsilon \rightarrow 0$ is relevant and from the analogue of equation (12) we find $T_0(\varepsilon) \sim -\frac{2}{\lambda} \log \varepsilon + \text{const}$. This logarithmic divergence can be interpreted as follows. For small ε the two legs of a loop need a certain minimal time in order to separate enough to enable the loop to close. This time can be estimated by requiring that $\varepsilon \exp(\lambda t/2)$ is of order 1, yielding the logarithmic dependence above. Substitution into (8) results in

$$\begin{aligned} K^{(2)}(\tau) &\sim \frac{8p^2 T^2}{mh\bar{d}(E)} \text{Re} \int_0^\infty d\varepsilon e^{\frac{ip^2 \varepsilon^2}{2mh\lambda}} \frac{2\varepsilon(\log \varepsilon + \text{const})}{\lambda \Sigma(E)} \\ &= \frac{16\tau^2}{\pi} \text{Re} \int_0^\infty d\varepsilon' e^{i\varepsilon'^2} \varepsilon' \log(\varepsilon'). \end{aligned} \quad (16)$$

Evaluating the real part of the last integral finally yields $K^{(2)}(\tau) \sim -2\tau^2$ in agreement with the τ^2 -term of the GOE form factor in (2).

In conclusion, we have shown that the off-diagonal contributions to the spectral form factor from two-loop orbits yield a τ^2 -term in agreement with random matrix theory. Its origin can be traced to properties of loops with small opening angle ε . It is expected that higher order terms in the expansion of $K(\tau)$ are related to multi-loop orbits, a point which is under investigation.

References

- [1] Bohigas O, Giannoni M J and Schmit C 1984 *Phys. Rev. Lett.* **52** 1
- [2] Haake F 1992 *Quantum Signatures of Chaos* (Berlin: Springer)
- [3] Hannay J H and Ozorio de Almeida A M 1984 *J. Phys. A: Math. Gen.* **17** 3429
Berry M V 1985 *Proc. R. Soc. A* **400** 229
- [4] Argaman N *et al* 1993 *Phys. Rev. Lett.* **71** 4326
- [5] Whitney R S, Lerner I V and Smith R A 1999 *Wave Random Media* **9** 179
- [6] Sieber M 2000 *J. Phys. A: Math. Gen.* **33** 6263
- [7] Sieber M and Richter K 2001 *Phys. Scr. T* **90** 128
- [8] Berkolaiko G, Schanz H and Whitney R S 2002 *Phys. Rev. Lett.* **88** 104101-1
- [9] Balazs N L and Voros A 1986 *Phys. Rep.* **143** 109
- [10] Parry W and Pollicott M 1990 *Astérisque* **187–188** 1
- [11] Argaman N, Imry Y and Smilansky U 1993 *Phys. Rev. B* **47** 4440
- [12] Aurich R and Steiner F 1988 *Physica D* **32** 451